ATMAM Mathematics Methods

Test 4 (2019) Calculator Free

SHENTON COLLEGE

Name: SOLUTIONS

Teacher: Friday Smith Ai

Time Allowed : 30 minutes

Marks	/30

Materials allowed: Formula Sheet.

Attempt all questions. Questions 1 to 5 are in this section.
All necessary working and reasoning must be show for full marks.
Marks may not be awarded for untidy or poorly arranged work.

For all questions, assume that the domain of x is restricted to ensure valid logarithms.

1 Differentiate with respect to x.

[3, 2, 1, 2] 8

a) $y = \ln\left(\dfrac{3x^2}{\sin x}\right)$

$y = \ln 3x^2 - \ln \sin x$ ✓ log law

$\dfrac{dy}{dx} = \dfrac{6x}{3x^2} - \dfrac{\cos x}{\sin x}$ ✓✓ differentiate

$= \dfrac{2}{x} - \dfrac{\cos x}{\sin x}$

b) $y = \ln\sqrt{(x^2-4)^3}$

$y = \ln(x^2-4)^{\frac{3}{2}}$ ✓ simplify & log law.

$= \frac{3}{2}\ln(x^2-4)$

$\dfrac{dy}{dx} = \dfrac{3}{2}\left(\dfrac{2x}{x^2-4}\right)$ ✓ differentiate.

$= \dfrac{3x}{x^2-4}$

c) $y = \ln\left(\dfrac{3}{5}\right)^2$

$\dfrac{dy}{dx} = 0$ ✓

d) $y = \ln\left(\dfrac{1}{x}\right)$

$\dfrac{dy}{dx} = \dfrac{\frac{-1}{x^2}}{\frac{1}{x}}$

$= \dfrac{-x}{x^2}$

$= -\dfrac{1}{x}$

or

$y = \ln x^{-1}$ ✓

$= -\ln x$

$\dfrac{dy}{dx} = -\dfrac{1}{x}$ ✓

2 Determine the following indefinite integrals.

a) $\int \frac{4}{3x} dx$ (2)

$= \frac{4}{3} \int \frac{3}{3x} dx$

$= \frac{4}{3} \ln|3x| + c$

✓ ln term

✓ coefficient.

b) $\int \frac{\sin x + \cos x}{\cos x - \sin x} dx$ (2)

$\frac{d}{dx}(\cos x - \sin x) = -\sin x - \cos x$

$- \int \frac{-\sin x - \cos x}{\cos x - \sin x} dx$

$= - \ln|\cos x - \sin x| + c$

✓ -ve sign

✓ ln term.

c) $\int \frac{x^2 + 2x + 1}{x^2 + 1} dx$ (3)

$= \int \frac{x^2+1}{x^2+1} + \frac{2x}{x^2+1} dx$

$= x + \ln|x^2+1| + c$

✓ split fraction

✓ x term

✓ ln term.

3 Evaluate the following definite integral, giving your answer as a **single logarithm**. (4)

$$\int_2^3 \frac{6x}{x^2-3}\,dx$$

$= 3\int_2^3 \frac{2x}{x^2-3}\,dx$ ✓ establish $\frac{f'(x)}{f(x)}$

$= 3\left[\ln|x^2-3|\right]_2^3$ ✓ integrate

$= 3(\ln 6 - \ln 1)$ ✓ substitute

$= 3\ln 6$ ✓ $\ln 1 = 0$ & simplify

4 If $f'(x) = \frac{x^2-3x+2}{x}$ and $f(2) = 2 + \ln 4$, determine the equation of $f(x)$. (5)

$f(x) = \int \frac{x^2-3x+2}{x}\,dx$ ✓ split fraction

$= \int x - 3 + \frac{2}{x}\,dx$ ✓ integrate

$= \frac{x^2}{2} - 3x + 2\ln|x| + c$ ✓ substitute $x = 2$

$f(2) = \frac{2^2}{2} - 3(2) + 2\ln 2 + c$ ✓ establish $\ln 4$ & solve for c

$2 + \ln 4 = 2 - 6 + \ln 4 + c$

$c = 6$ ✓ $f(x)$

$f(x) = \frac{x^2}{2} - 3x + 2\ln|x| + 6$

5 a) Differentiate $y = x^3 \ln x$ with respect to x. (2)

$$\frac{dy}{dx} = 3x^2 \ln x + x^3\left(\frac{1}{x}\right)$$

$$= 3x^2 \ln x + x^2$$

✓ product rule

✓ $\frac{d}{dx} \ln x$

b) Using your result from a), or otherwise, determine $\int x^2 \ln x \, dx$ (4)

$$\int x^2 \ln x \, dx$$

$$= \frac{1}{3} \int 3x^2 \ln x \, dx$$

$$= \frac{1}{3} \int 3x^2 \ln x + x^2 \, dx - \frac{1}{3} \int x^2 \, dx$$

$$= \frac{1}{3}(x^3 \ln x) - \frac{1}{9}x^3 + c$$

✓ coefficient to get $3x^2 \ln x$

✓ $+x^2 - x^2$

✓ FToC

✓ answer including extra term.

$$\int 3x^2 \ln x + x^2 \, dx = x^3 \ln x$$

$$3 \int x^2 \ln x \, dx + \int x^2 \, dx = x^3 \ln x$$

$$3 \int x^2 \ln x \, dx = x^3 \ln x - \int x^2 \, dx$$

$$\int x^2 \ln x \, dx = \frac{1}{3}\left(x^3 \ln x - \frac{x^3}{3} + c\right)$$

$$= \frac{x^3 \ln x}{3} - \frac{x^3}{9} + c$$

SHENTON COLLEGE

ATMAM Mathematics Methods

Test 4 (2019) Calculator Assumed

Name: ....SOLUTIONS......................................

Teacher: Friday Smith Ai

Time Allowed : 20 minutes

Marks	/18

Materials allowed: Classpad, Formula Sheet.

Attempt all questions. Questions 6 to 8 are in this section.
All necessary working and reasoning must be shown for full marks.
Marks may not be awarded for untidy or poorly arranged work.

6 Staff at the Local Government Policy Advice Committee have access to three different surveys about the number of people who would like to see street parking banned in all residential streets. Survey A involved 180 people, of whom 110 were in support of a ban. Survey B polled 275 people, with 162 favouring a ban. Survey C had 205 respondents, of which 125 agreed with the ban on street parking.

a) Given that the surveys were all carried out in almost identical conditions, which survey is likely to be the best representation of the proportion of people in the population who support a ban on street parking in residential streets? Briefly justify your choice. (2)

Survey B, due to the larger sample size.

b) The LGPAC decide to conduct another almost identical survey (since they haven't wasted all of their allocated budget yet), using a sample size of 150. Using your answer from a) as the best estimate for the population proportion, approximate the sampling distribution for samples of size 150. (3)

$\hat{p} = \frac{162}{275} \simeq 0.5891$ ✓ $\hat{p}$

$\sigma_{\hat{p}} = \sqrt{\frac{\frac{162}{275}\left(\frac{113}{275}\right)}{150}} \simeq 0.0402$ ✓ $\sigma_{\hat{p}}$

Sampling distribution for $n=150$ is $N(0.5891, 0.0402^2)$

✓ Normal with parameters

c) The additional survey of 150 people finds that 112 of them support the ban on street parking. After receiving these results, several of the LGPAC staff express concerns over the validity of the data. Use your mathematical knowledge to comment on whether the concerns of the staff might be justified. (3)

$$\frac{112}{150} \simeq 0.7467$$

✓ $\hat{p}$

use full values.

$$\frac{0.7467 - 0.5891}{0.0402} = 3.92$$

✓ z score

✓ interpret

This sample is 3.92 standard errors from the estimated population proportion, which is very unlikely.

7 In a game of chance, two six sided dice are rolled and the numbers shown on each dice are added together. If the result is a square number, the player wins, otherwise they lose that game.

a) A class of 30 students play the game 60 times each, recording the percentage of wins they get. Determine the distribution that could be used to model the percentage of wins that would be expected to occur if the game was played 60 times. Comment on the validity of this model. (3)

$$p = \frac{7}{36} \simeq 0.1944$$

✓ $\hat{p}$ & $\sigma_{\hat{p}}$

$$\sigma_{\hat{p}} = \sqrt{\frac{\frac{7}{36}\left(\frac{29}{36}\right)}{60}} \simeq 0.0511$$

✓ distribution

$$\hat{p} \sim N\left(\frac{7}{36}, 0.0511^2\right)$$

✓ shows validity

$$np = 11.67$$

$$n(1-p) = 48.33$$

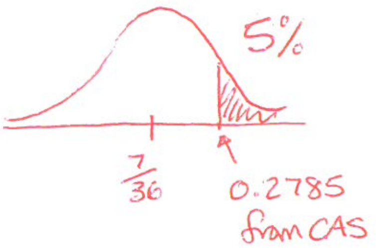
both are greater than 10, therefore Normal distribution is appropriate

b) Use the sampling distribution to estimate the probability that someone playing the game 60 times would win on more than 10 occasions? (1)

$$\hat{p} = \frac{10}{60} \simeq 0.1667$$

$$P\left(\hat{p} > \frac{10}{60}\right) = 0.7067$$

c) Consider the statement "At least 5% of people would be expected to win more than k games out of 60". Determine the largest integer value for k that makes the statement true. (2)



Proportion $= 0.2785$

$0.2785 \times 60 = 16.71$

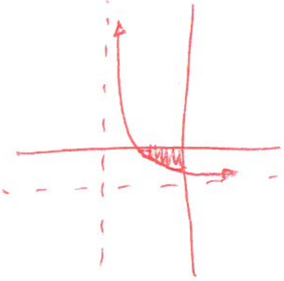
Largest value for k is 16

(since $k = 17$ would give less than 5%, not at least 5%)

✓ Proportion for 5% right tail.

✓ k correct.

8 Show use of calculus to determine the exact area trapped between the curve $y = \frac{1}{x+3} - 1$ and the axes. (4)



$$\int_{-2}^{0} \frac{1}{x+3} - 1 \, dx$$

✓ integral with correct bounds

$$= \Big[\ln|x+3| - x\Big]_{-2}^{0}$$

✓ integrate

$$= (\ln 3 - 0) - (\ln 1 + 2)$$

✓ substitute & simplify

$$= \ln 3 - 2$$

However as this integral is below the axis, we need to reverse the sign.

$$\therefore \text{Area} = 2 - \ln 3$$

✓ interpret for area